

Relation between state-distinction power and disturbance in quantum measurements

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Measurement of an informative observable strongly disturbs a quantum state. We study this so-called information-disturbance relation by introducing order relations based on the state-distinction power of an observable and the variety of nondisturbed observables of a channel. As a byproduct a novel quantitative inequality between compatible observable and channel is presented.

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I. INTRODUCTION

The concept of incompatibility¹ plays a crucial role in understanding the quantum theory. The position and momentum of a single particle are the most famous example of incompatible observables. While this example treats the incompatibility between two observables, the notion of incompatibility has been extended to general operations. Each class of operation in the quantum theory is characterized by its outcome space. An observable has a classical state space as its outcome space. An operation whose outcome is a quantum state is called a (quantum) channel. A pair of operations (possibly in different classes) is called incompatible if they cannot be realized simultaneously. That is, two operations are incompatible if there is no experimental device which contains these operations as its parts.

In this paper, we discuss the incompatibility between observables and channels. This kind of incompatibility has been studied extensively thus far. In fact, various conditions for an observable and a channel to be compatible have been obtained²⁻⁷. A clue to study the conditions for the (in)compatibility is a so-called information disturbance relation: “strong” disturbance is inevitable in measuring “informative” observables. By introducing quantitative measures for “informative” character of an observable and “disturbing” character of a channel, one elaborate inequalities between them, which give concrete limitations on the (in)compatible operations. While such quantitative inequalities play essential roles in practical problems such as quantum cryptography⁸, they have a common drawback. In fact each quantitative measure on the observable space (channel space) allows any observables (resp. channels) to be comparable. Thus the observable space (channel space) is made linearly ordered and its structure is spoiled to some extent. Recently one of the authors proposed a qualitative (structural) representation of information disturbance relation⁹. An observable A (a channel Λ_1) is defined to be more informative (resp. less disturbing) than B (resp. Λ_2) if and only if B (resp. Λ_2) can be obtained by A (resp. Λ_1) followed by a post-processing. This definition makes the observable space (channel space) a preordered space which is not linearly ordered.

In this paper, we study another qualitative representation of information disturbance relation. Each operation space has a rich structure and the order structure introduced in the prior study⁹ is not the unique one. We employ an order in the observable space determined by the state-distinction power discussed in^{10,11}. An order relation in the channel space

is introduced by focusing on the invariant observables, which has a natural interpretation in terms of the quantum non-demolition measurement^{12–14}. We show that these order relations are related to each other by an information disturbance trade-off relation. In addition, we present a quantitative information disturbance trade-off relation motivated by this qualitative relation.

The rest of the paper is organized as follows: In section II the basic notions used throughout the paper are introduced. In section III, we introduce order relations in the observable space and the channel space. Section IV discusses a trade-off relation in terms of the order structure. In section V we discuss a quantitative tradeoff relation between measures characterizing state-distinction power and nondisturbance.

II. PRELIMINARIES

Let \mathcal{H} be a finite-dimensional Hilbert space. We denote by $\mathcal{B}(\mathcal{H})$ the set of (bounded) operators on \mathcal{H} . A state is represented by a positive operator with unit trace (i.e., a density operator). We denote the set of density operators by $\mathcal{S}(\mathcal{H})$. In this paper, we treat only observables with at most countable outcome sets. An observable whose outcome set is Ω is represented by a family of positive operator $\mathbf{A} = \{\mathbf{A}(x)\}_{x \in \Omega}$ satisfying $\sum_{x \in \Omega} \mathbf{A}(x) = \mathbb{1}$ which is called a positive-operator-valued measure (POVM). Suppose that we prepare a state ρ and measure an observable \mathbf{A} . Then we obtain each outcome $x \in \Omega$ with a probability $\text{tr}[\rho \mathbf{A}(x)]$. Thus each observable defines an affine map from the set of the states $\mathcal{S}(\mathcal{H})$ to the set of probability distributions on its outcome set $\{\{p(x)\}_{x \in \Omega} \mid p(x) \geq 0, \sum_x p(x) = 1\}$. While observable is sufficient to describe statistical aspects of the classical outcomes of a measurement process, we are often required to treat a quantum state after a measurement process. This state change is described by a map whose output space is a set of quantum states. We call a map from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{K})$ (\mathcal{K} is an output Hilbert space) a channel if it is affine and its natural extension to $\mathcal{S}(\mathcal{H} \otimes \mathbb{C}^N)$ has its codomain in $\mathcal{S}(\mathcal{K} \otimes \mathbb{C}^N)$ for each $N \in \mathbb{N}$. This map can be linearly extended to $\mathcal{B}(\mathcal{H})$. In this paper, we treat channels whose output system coincides with the input system. That is, $\mathcal{K} = \mathcal{H}$ is assumed. It is well known that each channel can be represented by an operator sum form (Kraus representation)¹⁵. There exists, for a given channel Λ , a family of operators $\{K_n\} \subset \mathcal{B}(\mathcal{H})$ satisfying $\Lambda(\rho) = \sum_n K_n \rho K_n^*$, where each K_n is called a Kraus operator and satisfies $\sum_n K_n^* K_n = \mathbb{1}$. Let us emphasize that

the Kraus representation is not uniquely determined for a given channel. In fact, each Kraus representation can be regarded to describe both classical and quantum outputs. A Kraus operators $\{K_n\}$ defines a POVM $\{K_n^*K_n\}$ and a channel $\Lambda(\rho) = \sum_n K_n \rho K_n^*$. Therefore an observable $\{K_n^*K_n\}_n$ and a channel $\Lambda(\rho) = \sum_n K_n \rho K_n^*$ are compatible. For a fixed channel Λ , any post-processing of the classical outcome does not spoil the compatibility. Thus for any stochastic matrix M_{xn} ($M_{xn} \geq 0, \sum_x M_{xn} = 1$ for each n), an observable defined by $\{\sum_n M_{xn} K_n^* K_n\}_x$ is also compatible with Λ . One can show that a family of operators defined by $\hat{K}_{x,n} := \sqrt{M_{xn}} K_n$ gives another Kraus representation of Λ as $\Lambda(\rho) = \sum_x \sum_n \hat{K}_{x,n}^* \rho \hat{K}_{x,n}$. Thus we arrive at a convenient representation of the compatibility. An observable \mathbf{A} and a channel Λ are compatible if and only if there exist a set of Kraus operators $\{K_{x,i}\}$ such that $\mathbf{A}(x) = \sum_i K_{x,i}^* K_{x,i}$ and $\Lambda(\rho) = \sum_x \sum_i K_{x,i} \rho K_{x,i}^*$ for any state $\rho \in \mathcal{S}(\mathcal{H})$ hold. (See¹⁶ for the detail.)

III. PREORDER OF OBSERVABLE AND CHANNEL

Our aim is to study the information-disturbance tradeoff between observables and channels without introducing any quantitative measures. The language of order theory is appropriate for this purpose. In this section, we introduce order structures on the observable space and the channel space.

A. Order structure of observables induced by state-distinction power

A pair of states ρ_1 and ρ_2 is called distinguishable by observable \mathbf{A} if and only if there exists an outcome $x \in \Omega$ such that $\text{tr}[\rho_1 \mathbf{A}(x)] \neq \text{tr}[\rho_2 \mathbf{A}(x)]$; see Fig 1. By using this distinguishability, a relation between observables, called state-distinction power is introduced^{10,11}.

Definition 1. An observable \mathbf{A} has larger state-distinction power than observable \mathbf{B} if and only if any pair of states distinguishable by \mathbf{B} is also distinguishable by \mathbf{A} . We denote this relation by $\mathbf{A} \succsim_i \mathbf{B}$.

It is easy to see that the state-distinction power is a preorder. That is, the relation is reflexive ($\mathbf{A} \succsim_i \mathbf{A}$ for any \mathbf{A}) and transitive ($\mathbf{A} \succsim_i \mathbf{B}$ and $\mathbf{B} \succsim_i \mathbf{C}$ implies $\mathbf{A} \succsim_i \mathbf{C}$). On the other hand, this relation does not satisfy antisymmetric property and is not a partial order. (See an argument below.)

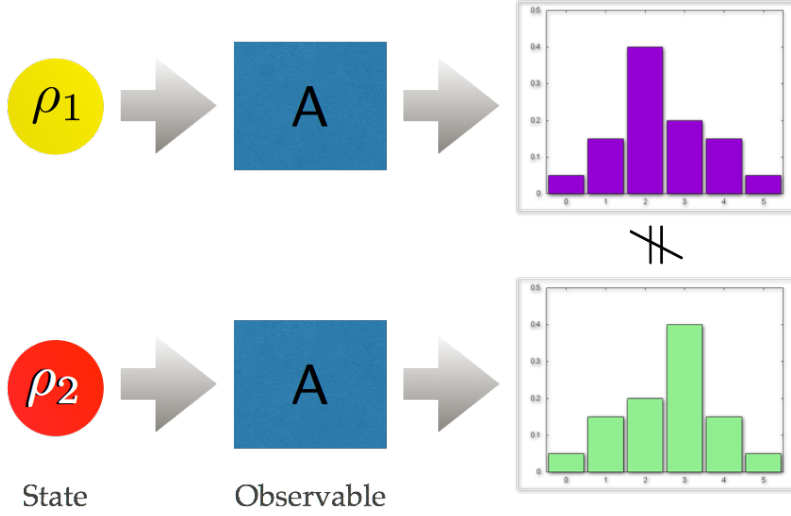


FIG. 1. By measuring an observable A , we discriminate a pair of state ρ_1 and ρ_2 from its probability distribution.

We emphasize that this relation is not total. That is, there is a pair of observables A and B which satisfies neither $A \precsim_i B$ nor $B \precsim_i A$. For instance, in a qubit system ($\mathcal{H} = \mathbb{C}^2$), sharp observables determined by Pauli matrices σ_x , σ_y and σ_z have no relation in state-distinction power, e.g. $\sigma_x \not\precsim \sigma_y$ and $\sigma_x \not\precsim \sigma_z$. In the sequel, let us consider a qutrit system ($\mathcal{H} = \mathbb{C}^3$). Also in a qutrit system ($\mathcal{H} = \mathbb{C}^3$), sharp observables determined by components of spin,

$$S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (1)$$

are also incomparable with each other.

We introduce linear span of POVM A by

$$\mathcal{L}(A) = \left\{ X \in \mathcal{B}(\mathcal{H}) \mid X = \sum_x c_x A(x), c_x \in \mathbb{C} \right\}. \quad (2)$$

The following observation gives a simple criterion for a pair of observables to be comparable.

Lemma 1. Let A and B be observables. The following conditions are equivalent.

- (i) $A \precsim_i B$.
- (ii) $\mathcal{L}(A) \subset \mathcal{L}(B)$.

Proof. (ii) \Rightarrow (i). For each $A(x)$, there exists coefficients d_{xy} such that $A(x) = \sum_y d_{xy} B(y)$. Suppose that ρ_1 and ρ_2 are distinguishable by A , that is $\text{tr}[(\rho_1 - \rho_2)A(x)] \neq 0$. The right hand side is equal to $\text{tr}[(\rho_1 - \rho_2) \sum_y d_{xy} B(y)] = \sum_y d_{xy} \text{tr}[(\rho_1 - \rho_2)B(y)]$. There exists $y \in \Omega_B$ such that $\text{tr}[(\rho_1 - \rho_2)B(y)] \neq 0$.

(i) \Rightarrow (ii). Assume that $\mathcal{L}(A) \not\subseteq \mathcal{L}(B)$. There exists $x \in \Omega_A$ such that $\left\{ \{ B(y) \}_{y \in \Omega_B}, A(x) \right\}$ is linearly independent. By using Gram-Schmidt orthonormalization of Hilbert-Schmidt inner product defined by $\langle A, B \rangle = \text{tr}[A^* B]$, the orthonormalized set $\left\{ \{ C(y) \}_{y \in \Omega}, C(0) \right\}$ is obtained. Note that by construction the linear hull of $\{ B(y) \}_{y \in \Omega_B}$ coincides with that of $\{ C(y) \}_{y \in \Omega}$. Thus it follows that for every $y \in \Omega_B$ $\langle B(y), C(0) \rangle = 0$ and also $\langle B(y), C(0)^* \rangle = 0$. One can assume that $C(0)$ is self-adjoint. Since $\sum_y B(y) = \mathbb{1}$, $\mathbb{1}$ and $C(0)$ are orthogonal. Therefore, $\text{tr}[C(0)] = 0$ holds.

Let d be a dimension of Hilbert space \mathcal{H} and $\rho_0 = \mathbb{1}/d$ be a completely mixed state. For sufficiently small $|\epsilon|$, an operator $\rho_\epsilon = \rho_0 + \epsilon C(0)$ is also state because $C(0)$ is traceless. $C(0)$ is orthogonal to $B(y)$ for any $y \in \Omega_B$, $\text{tr}[\rho_0 C(0)] = \text{tr}[\rho_\epsilon C(0)]$ holds. This implies that these states are not distinguishable by B . On the other hand, since $A(x)$ is not orthogonal to $C(0)$, $\text{tr}[\rho_0 A(x)] \neq \text{tr}[\rho_\epsilon A(x)]$ holds, that is ρ_0 and ρ_ϵ are distinguishable by A . This contradicts the assumption $A \prec_i B$. \square

An observable A is called informationally complete if $\text{tr}[\rho_1 A(x)] = \text{tr}[\rho_2 A(x)]$ for all x implies $\rho_1 = \rho_2$. Thus by definition it is obvious that each informationally complete observable is a maximal element. In fact, the linear span of informationally complete observables coincides with $\mathcal{B}(\mathcal{H})$ ¹⁷.

Example 1. On the qubit system, a symmetric informationally complete (SIC) POVM¹⁸ A_{SIC} is given by

$$\begin{aligned} A_{\text{SIC}}(0) &= \frac{1}{4} \begin{pmatrix} 1 + \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} - i\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{3}} & 1 - \frac{1}{\sqrt{3}} \end{pmatrix}, & A_{\text{SIC}}(1) &= \frac{1}{4} \begin{pmatrix} 1 - \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} - i\frac{1}{\sqrt{3}} & 1 + \frac{1}{\sqrt{3}} \end{pmatrix}, \\ A_{\text{SIC}}(2) &= \frac{1}{4} \begin{pmatrix} 1 - \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} - i\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{3}} & 1 + \frac{1}{\sqrt{3}} \end{pmatrix}, & A_{\text{SIC}}(3) &= \frac{1}{4} \begin{pmatrix} 1 + \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} + i\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} - i\frac{1}{\sqrt{3}} & 1 - \frac{1}{\sqrt{3}} \end{pmatrix}. \end{aligned}$$

It is easy to check that their linear span coincides with $\mathcal{B}(\mathcal{H})$.

Example 2. A SIC-POVM on a qutrit system is given by

$$\begin{aligned}
A_{\text{SIC}}^{\text{qutrit}}(0) &= \frac{1}{6} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & A_{\text{SIC}}^{\text{qutrit}}(1) &= \frac{1}{6} \begin{pmatrix} 1 & \omega^* & 0 \\ \omega & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & A_{\text{SIC}}^{\text{qutrit}}(2) &= \frac{1}{6} \begin{pmatrix} 1 & \omega & 0 \\ \omega^* & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
A_{\text{SIC}}^{\text{qutrit}}(3) &= \frac{1}{6} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, & A_{\text{SIC}}^{\text{qutrit}}(4) &= \frac{1}{6} \begin{pmatrix} 1 & 0 & \omega^* \\ 0 & 0 & 0 \\ \omega & 0 & 1 \end{pmatrix}, & A_{\text{SIC}}^{\text{qutrit}}(5) &= \frac{1}{6} \begin{pmatrix} 1 & 0 & \omega \\ 0 & 0 & 0 \\ \omega^* & 0 & 1 \end{pmatrix}, \\
A_{\text{SIC}}^{\text{qutrit}}(6) &= \frac{1}{6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, & A_{\text{SIC}}^{\text{qutrit}}(7) &= \frac{1}{6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \omega^* \\ 0 & \omega & 1 \end{pmatrix}, & A_{\text{SIC}}^{\text{qutrit}}(8) &= \frac{1}{6} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \omega \\ 0 & \omega^* & 1 \end{pmatrix}
\end{aligned}$$

where $\omega = \exp\left(\frac{2\pi}{3}i\right)$. It is easy to see that their linear span coincides with $\mathcal{B}(\mathbb{C}^3)$.

On the other hand, a minimal element of this preorder is a trivial observable which is written as $A(x) = p_x \mathbb{1}$ with $p_x \geq 0$ for each $x \in \Omega_x$. Any pair of states ρ_1 and ρ_2 are not distinguishable by a trivial observable since $\text{tr}[\rho_1 A(x)] = \text{tr}[\rho_2 A(x)] = p_x$ holds. It is easy to see that every minimal element is a trivial observable.

B. Order structure of channels induced by nondisturbing measurement

Let us introduce an order structure of channel space in the light of disturbance property. A channel describes a state change from ρ to $\Lambda(\rho)$. Suppose that we examine the change by measuring an observable $B = \{B(y)\}_{y \in \Omega}$. We can confirm that ρ and $\Lambda(\rho)$ are different if the probability distributions $\{\text{tr}[\rho B(y)]\}$ and $\{\text{tr}[\Lambda(\rho) B(y)]\}$ are distinct. On the other hand, if we choose B so that $\text{tr}[\rho B(y)] = \text{tr}[\Lambda(\rho) B(y)]$ holds for all $y \in \Omega$ and for any input state ρ , this B is useless to study the state change. Motivated by this observation, we call Λ nondisturbing¹⁹ for an observable B (or B is not disturbed by Λ) if $\text{tr}[\rho B(y)] = \text{tr}[\Lambda(\rho) B(y)]$ holds for any input state ρ and $y \in \Omega$; see Fig 2.

We introduce an order structure on the set of channels.

Definition 2. Let Λ_1, Λ_2 be channels. If any observable which is not disturbed by Λ_1 is neither disturbed by Λ_2 , then we call Λ_2 less disturbing than Λ_1 and denote by $\Lambda_1 \preceq_f \Lambda_2$.

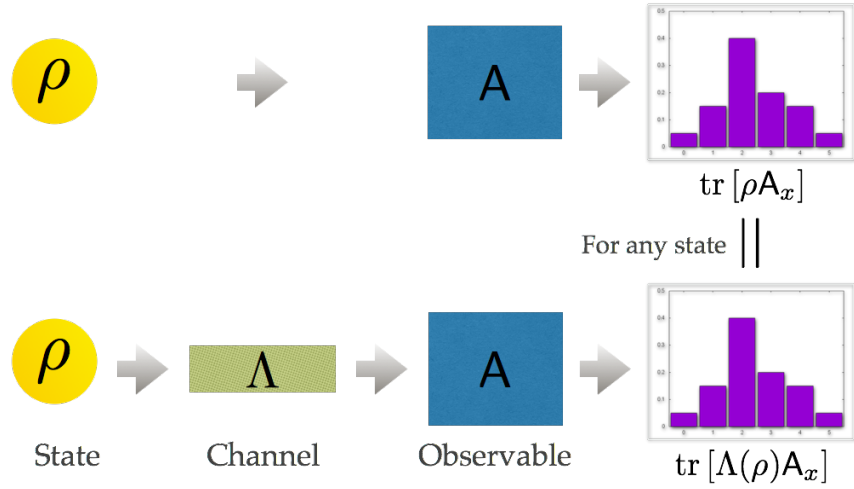


FIG. 2. A channel Λ does not disturb the output from measuring an observable A

This binary relation is preorder. The Heisenberg picture is convenient to characterize this preorder. A channel Λ has a dual description (i.e., Heisenberg picture) $\Lambda^* : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ defined by $\text{tr}[\Lambda(\rho)X] = \text{tr}[\rho\Lambda^*(X)]$ for any ρ . The set of fixed points of the map Λ^* is defined by

$$\text{Fix}(\Lambda^*) = \{ X \in \mathcal{B}(\mathcal{H}) \mid \Lambda^*(X) = X \}, \quad (3)$$

which forms a subspace of $\mathcal{B}(\mathcal{H})$. The following property is easy to verify.

Lemma 2. Let Λ_1 and Λ_2 be a channel. The following conditions are equivalent.

- (i) $\Lambda_1 \lesssim_f \Lambda_2$.
- (ii) $\text{Fix}(\Lambda_1^*) \subset \text{Fix}(\Lambda_2^*)$.

Thus there exists the greatest element in the channel space. The greatest element is an identity channel id . On the other hand, the least element does not exist. One of the minimal elements is a channel whose output state does not depend on the input state. That is, Λ has a fixed ρ_0 such that $\Lambda(\rho) = \rho_0$ holds for any ρ . We will see later other examples of minimal elements.

IV. INFORMATION-DISTURBANCE TRADEOFF RELATION

In this section, we study a qualitative information-disturbance trade-off relation based on the preorder structure introduced in the last section. We examine the maximal elements in the observable space.

Theorem 1. Let \mathbf{A} be an informationally complete observable. For a channel $\Lambda_{\mathbf{A}}$ compatible with \mathbf{A} , it holds that

$$\text{Fix}(\Lambda_{\mathbf{A}}^*) = \mathbb{C}\mathbf{1}.$$

That is, $\Lambda_{\mathbf{A}}^*(B) = B$ implies $B = c\mathbf{1}$ with some complex number c .

We call that an observable $\mathbf{A} = \{\mathbf{A}\}_{x \in \Omega}$ perfectly distinguishes a pair of states ρ_0 and ρ_1 if there exists $O \subset \Omega$ satisfying $\sum_{x \in O} \text{tr}[\rho_0 \mathbf{A}(x)] = \sum_{x \in \Omega \setminus O} \text{tr}[\rho_1 \mathbf{A}(x)] = 1$. To prove Theorem 1, we use following lemma.

Lemma 3 (Example 3.61 in¹⁷). Informationally complete observables cannot distinguish perfectly any pair of states.

Proof of Theorem 1. Let B be an element of $\text{Fix}(\Lambda_{\mathbf{A}}^*)$. As B^* is also invariant with $\Lambda_{\mathbf{A}}$, $\text{Re}(B) := \frac{B+B^*}{2}$ and $\text{Im}(B) := \frac{B-B^*}{2i}$ are elements of $\text{Fix}(\Lambda_{\mathbf{A}}^*)$. We assume that B is not trivial. That is, it is not proportional to $\mathbf{1}$. By linearity and unitality of $\Lambda_{\mathbf{A}}$, we can choose B is positive and $\|B\| = 1$ and $\|1 - B\| = 1$. Let $|\psi_1\rangle$ and $|\psi_0\rangle$ be a normalized eigenvector such that $B|\psi_1\rangle = |\psi_1\rangle$ and $B|\psi_0\rangle = 0$.

Let \mathbf{B} be an observable $\mathbf{B}(0) = B$ and $\mathbf{B}(1) = \mathbf{1} - B$. Using this observable, a pair of states $\{|\psi_1\rangle\langle\psi_1|, |\psi_0\rangle\langle\psi_0|\}$ is perfectly distinguishable. Since $\Lambda_{\mathbf{A}}$ and \mathbf{B} are compatible, \mathbf{A} and \mathbf{B} are jointly measurable. Since \mathbf{A} is an informationally complete and \mathbf{B} can distinguish perfectly some pair of states, the joint observable of them is an informationally complete and can distinguish perfectly some pair of states. However, it contradicts the Lemma 3. \square

Corollary 1. A channel $\Lambda_{\mathbf{A}}$ which is compatible with an informationally complete observable \mathbf{A} has only one invariant state.

Proof. This follows from Theorem 1 and the fact $\dim \text{Fix}(\Lambda) = \dim \text{Fix}(\Lambda^*)$. \square

Theorem 1 implies that channels compatible with a maximal observable must be minimum in the channel space. In other words, for each informationally complete observable \mathbf{A} , any

channel compatible with \mathbf{A} is a maximal element in the channel space. Let us remark that the converse is not true. That is, there exists a minimal channel which is not compatible with any informationally complete observable (see Example 3). To investigate this property systematically, we introduce a subclass of channels. A channel is said to satisfy property F if and only if it has a faithful invariant state. That is, Λ with property F has a state ρ_0 whose kernel is $\{0\}$ such that $\Lambda(\rho_0) = \rho_0$. Fixed point $\text{Fix}(\Lambda^*)$ of this channel forms a subalgebra of $\mathcal{B}(\mathcal{H})^{20}$. By using a Kraus decomposition $\Lambda^*(\cdot) = \sum_i K_i^* \cdot K_i$, its fixed point $\text{Fix}(\Lambda^*)$ is represented as

$$\text{Fix}(\Lambda^*) = \{ K_i, K_i^* \}_i' = \{ X \in \mathcal{B}(\mathcal{H}) \mid [X, K_i] = [X, K_i^*] = 0 \} \quad (4)$$

Thus $\text{Fix}(\Lambda^*)$ forms an algebra.

Lemma 4. Let \mathbf{A} be an observable and Λ be a channel compatible with \mathbf{A} . If Λ has property F, then

$$\text{Fix}(\Lambda^*) \subset \{ \mathbf{A}(x) \}_{x \in \Omega_{\mathbf{A}}} \quad (5)$$

holds.

Proof. By using a Kraus decomposition $\Lambda^*(\cdot) = \sum_{x,j} K_{x,j}^* \cdot K_{x,j}$, an observable \mathbf{A} compatible with Λ satisfies $\mathbf{A}(x) = \sum_j K_{x,j}^* K_{x,j}$. For $X \in \mathcal{B}(\mathcal{H})$, it holds

$$[\mathbf{A}(x), X] = \sum_j [K_{x,j}^* K_{x,j}, X] = \sum_j K_{x,j}^* [K_{x,j}, X] + \sum_j [K_{x,j}^*, X] K_{x,j} = 0. \quad (6)$$

□

Let us consider a Lüders channel $\Lambda_{\mathbf{A}}^L(\cdot) := \sum_x \sqrt{\mathbf{A}(x)} \cdot \sqrt{\mathbf{A}(x)}$. The completely mixed state $\rho_0 = \frac{\mathbb{1}}{\dim \mathcal{H}}$ is an invariant state of the Lüders channel. For $X \in \{ \mathbf{A}(x) \}_{x \in \Omega_{\mathbf{A}}}'$, it follows that $\Lambda_{\mathbf{A}}^L(X) = \sum_x \sqrt{\mathbf{A}(x)} X \sqrt{\mathbf{A}(x)} = X$. Thus for the Lüders channel we have $\text{Fix}(\Lambda_{\mathbf{A}}^{L*}) = \{ \mathbf{A}(x) \}_{x \in \Omega_{\mathbf{A}}}'$. It means that the Lüders channel is one of the minimally disturbing channel among channel satisfying property F compatible with \mathbf{A} .

We have the following information-disturbance relation among Lüders channels.

Theorem 2. Let \mathbf{A} and \mathbf{B} be observables, and $\Lambda_{\mathbf{A}}^L$ and $\Lambda_{\mathbf{B}}^L$ be their corresponding Lüders channel. If $\mathbf{A} \succsim_i \mathbf{B}$ holds, $\Lambda_{\mathbf{A}}^L \precsim_f \Lambda_{\mathbf{B}}^L$ is satisfied.

Proof. From Lemma 1 $\mathcal{L}(\mathbf{A}) \supset \mathcal{L}(\mathbf{B})$ holds. It follows that $\{\mathbf{A}(x)\}'_{x \in \Omega_A} \subset \{\mathbf{B}(y)\}'_{y \in \Omega_B}$. Thus we conclude $\Lambda_A^L \lesssim_f \Lambda_B^L$ due to Lemma 4. □

$\{\mathbf{A}(x)\}'_{x \in \Omega_A} \subset \{\mathbf{B}(y)\}'_{y \in \Omega_B}$ does not imply $\mathcal{L}(\mathbf{A}) \supset \mathcal{L}(\mathbf{B})$. Therefore converse of the above theorem does not hold in general. We illustrate this in the following qubit example. Let $|0\rangle$ and $|1\rangle$ be an orthonormal basis of \mathbb{C}^2 . Let $|x_{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and $|y_{\pm}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. Observables \mathbf{A}^{ZX} and \mathbf{B}^{ZY} are introduced by $\mathbf{A}^{ZX}(0) = \frac{1}{2}|0\rangle\langle 0|$, $\mathbf{A}^{ZX}(1) = \frac{1}{2}|1\rangle\langle 1|$, $\mathbf{A}^{ZX}(2) = \frac{1}{2}|x_+\rangle\langle x_+|$, $\mathbf{A}^{ZX}(3) = \frac{1}{2}|x_-\rangle\langle x_-|$ and $\mathbf{B}^{ZY}(0) = \frac{1}{2}|0\rangle\langle 0|$, $\mathbf{B}^{ZY}(1) = \frac{1}{2}|1\rangle\langle 1|$, $\mathbf{B}^{ZY}(2) = \frac{1}{2}|y_+\rangle\langle y_+|$, $\mathbf{B}^{ZY}(3) = \frac{1}{2}|y_-\rangle\langle y_-|$. It is easy to see $\{\mathbf{A}^{ZX}(x)\}'_{x \in \Omega_A} = \{\mathbf{B}^{ZY}(y)\}'_{y \in \Omega_B} = \mathbb{C}\mathbf{1}$ and $\mathcal{L}(\mathbf{A}^{ZX}) \neq \mathcal{L}(\mathbf{B}^{ZY})$.

Example 3. Let us suppose $\dim \mathcal{H} = N \geq 3$, which has a standard basis $\{|n\rangle\}_{n=0}^{N-1}$. We define unitary matrices $U = \sum_n e^{i\frac{2\pi n}{N}}|n\rangle\langle n|$ and $V = \sum_n |n+1\rangle\langle n|$, where $|N\rangle$ is identified with $|0\rangle$. It is easy to see that $\{U, V\}' = \mathbb{C}\mathbf{1}$ holds. We consider a channel $\Lambda(\rho) = \frac{1}{2}U\rho U^* + \frac{1}{2}V\rho V^*$. This has a faithful fixed point $\frac{\mathbf{1}}{N}$. Due to the previous discussion, we conclude that this channel is a minimal element in the channel space. On the other hand, every POVM element of an observable compatible with this channel is spanned by $\{\mathbf{1}, U^*V, V^*U\}$. Thus for any \mathbf{A} compatible with Λ , $\mathcal{L}(\mathbf{A}) \leq 3$ holds. We can conclude that no informationally complete observable is compatible.

In the paper²¹, the problem of sequential measurement has been discussed in the light of post-processing order structure. Inspired by this, we consider the following problem. Suppose that we first measure an observable \mathbf{A} and then measure an observable \mathbf{B} which was not disturbed by the first measurement. This sequential measurement gives a joint measurement of \mathbf{A} and \mathbf{B} . Is it possible to measure \mathbf{A} so that this joint measurement gives a maximal state distinction power? In other words, when the joint observable \mathbf{A} and \mathbf{B} can be informationally complete? A trivial \mathbf{A} gives a trivial example. As measurement of \mathbf{A} does not cause any disturbance, one can measure any informationally complete observable \mathbf{B} afterwards. A less trivial example is as follows. We assume that \mathcal{H} has a tensor product structure as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Suppose that \mathbf{A} is an informationally complete observable on \mathcal{H}_1 and \mathbf{B} is also another informationally complete observable on \mathcal{H}_2 . Then $\mathbf{A} \otimes \mathbf{B}$ becomes an informationally complete observable on \mathcal{H} . We can construct a channel compatible with \mathbf{A}

acting on only \mathcal{H}_1 , which naturally does not disturb \mathbf{B} . This “preservation of state distinction power” does not hold in general.

Theorem 3. Let \mathbf{A} be an observable such that $\{\mathbf{A}(x)\}'' \cap \{\mathbf{A}(x)\}'$ is not trivial. That is, this algebra is strictly larger than $\mathbb{C}\mathbf{1}$. We measure this \mathbf{A} by a compatible channel Λ . Let \mathbf{B} be an observable satisfying $\mathbf{B}(y) \in \text{Fix}(\Lambda^*)$ for all $y \in \Omega_{\mathbf{B}}$. The subsequent measurement \mathbf{B} defines a joint measurement of \mathbf{A} and \mathbf{B} . This joint measurement cannot be informationally complete.

Proof. Let Λ be a channel compatible with \mathbf{A} . We denote its Kraus representation $\Lambda(\cdot) = \sum_x \sum_i K_{x,i}^* \cdot K_{x,i}$. $\text{Fix}(\Lambda^*)$ is contained in $\{\mathbf{A}(x)\}'$.

As $\{\mathbf{A}(x)\}'$ is a finite von Neumann algebra, it can be represented as

$$\{\mathbf{A}(x)\}' = \bigoplus_{n=1,2,\dots,N} \mathcal{B}(\mathcal{H}_n) \otimes \mathbb{1}_{\mathcal{K}_n},$$

where $\mathcal{H}_n \otimes \mathcal{K}_n$ are orthogonal subspace satisfying $\bigoplus \mathcal{H}_n \otimes \mathcal{K}_n = \mathcal{H}$. Therefore $\{\mathbf{A}(x)\}'' = \bigoplus_{n=1,2,\dots,N} \mathbb{1}_{\mathcal{H}_n} \otimes \mathcal{B}(\mathcal{K}_n)$ and $\{\mathbf{A}(x)\}'' \cap \{\mathbf{A}(x)\}' = \bigoplus_n \mathbb{C}P_n$ hold where P_n is a projection on $\mathcal{H}_n \otimes \mathcal{K}_n$. The assumption of its nontriviality is equivalent to a condition $N \geq 2$.

Let us construct states which cannot be distinguished by the joint observable \mathbf{A} and \mathbf{B} . We define a unitary operator U by $U = \bigoplus_n e^{i\theta_n} P_n$ satisfying $\theta_n \neq \theta_m \pmod{2\pi}$ for all $n \neq m$. The states we are now considering are a some arbitrary state $|\phi_0\rangle$ and $|\phi_1\rangle = U|\phi_0\rangle$. These states cannot be distinguished by the joint measurement. Actually,

$$\begin{aligned} \Pr(\mathbf{A} = x, \mathbf{B} = y | \phi_0) &= \left\langle \psi_0 \left| \sum_i K_{x,i}^* \mathbf{B}(y) K_{x,i} \psi_0 \right. \right\rangle \\ &= \langle \psi_0 | \mathbf{A}(x) \mathbf{B}(y) \psi_0 \rangle \\ \Pr(\mathbf{A} = x, \mathbf{B} = y | \phi_1) &= \left\langle \psi_0 \left| U^* \sum_i K_{x,i}^* \mathbf{B}(y) K_{x,i} U \psi_0 \right. \right\rangle \\ &= \langle \psi_0 | U^* \mathbf{A}(x) \mathbf{B}(y) U \psi_0 \rangle \\ &= \langle \psi_0 | \mathbf{A}(x) \mathbf{B}(y) \psi_0 \rangle. \end{aligned}$$

Thus they correspond to each other. However, $|\psi_0\rangle$ is not same $|\psi_1\rangle$ as a state in general. \square

V. QUANTITATIVE INEQUALITY

We consider a quantitative version of our result. We note that $\mathcal{L}(\mathbf{A})$ and $\text{Fix}(\Lambda^*)$ are vector spaces. One of the natural quantities measuring the size of a complex vector space is its dimension. We have the following theorem.

Theorem 4. For an observable \mathbf{A} and a channel Λ^* which are compatible, it holds

$$\dim \mathcal{L}(\mathbf{A}) + \dim \text{Fix}(\Lambda^*) \leq (\dim \mathcal{H})^2 + 1. \quad (7)$$

Proof. We first assume that the channel Λ satisfies property F. We prove $\Delta := (\text{RHS}) - (\text{LHS}) \geq 0$.

$$\begin{aligned} \Delta &= (\dim \mathcal{H})^2 + 1 - \dim \mathcal{L}(\mathbf{A}) - \dim \text{Fix}(\Lambda^*) \\ &\geq (\dim \mathcal{H})^2 + 1 - \dim \mathcal{L}(\mathbf{A}) - \dim \mathcal{L}(\mathbf{A})' \quad (\because \text{Lemma 4}) \\ &\geq (\dim \mathcal{H})^2 + 1 - \dim \mathcal{L}(\mathbf{A})'' - \dim \mathcal{L}(\mathbf{A})' \end{aligned}$$

By using notation of proof of Theorem 3,

$$\begin{aligned} \Delta &\geq (\dim \mathcal{H})^2 + 1 - \dim \oplus_n \mathbb{1}_{\mathcal{H}_n} \otimes \mathcal{B}(\mathcal{K}_n) - \dim \oplus_n \mathcal{B}(\mathcal{H}_n) \otimes \mathbb{1}_{\mathcal{K}_n} \\ &= (\dim \mathcal{H})^2 + 1 - \sum_n (\dim \mathcal{K}_n)^2 - \sum_n (\dim \mathcal{H}_n)^2 \\ &= \left(\sum_n (\dim \mathcal{H}_n) \times (\dim \mathcal{K}_n) \right)^2 + 1 - \sum_n ((\dim \mathcal{H}_n)^2 + (\dim \mathcal{K}_n)^2) \quad (\because \mathcal{H} = \oplus_n \mathcal{H}_n \otimes \mathcal{K}_n) \\ &\geq \sum_n (\dim \mathcal{H}_n)^2 (\dim \mathcal{K}_n)^2 + \binom{N}{2} + 1 - \sum_n ((\dim \mathcal{H}_n)^2 + (\dim \mathcal{K}_n)^2) \end{aligned}$$

Since an inequality $\binom{N}{2} + 1 - N = \frac{1}{2}(N-1)(N-2) \geq 0$ holds for $N \geq 1$,

$$\begin{aligned} \Delta &\geq \sum_n (\dim \mathcal{H}_n)^2 (\dim \mathcal{K}_n)^2 + N - \sum_n ((\dim \mathcal{H}_n)^2 + (\dim \mathcal{K}_n)^2) \\ &= \sum_n ((\dim \mathcal{H}_n)^2 - 1)((\dim \mathcal{K}_n)^2 - 1) \\ &\geq 0 \end{aligned}$$

We discuss the case Λ does not satisfy property F. We use a support projection operator P_Λ introduced by Lindblad²⁰. This projection operator P_Λ is defined by the smallest projection satisfying $\text{tr}[\rho P_\Lambda] = 1$ for all $\rho \in \mathcal{S}(\mathcal{H}) \cap \text{Fix}(\Lambda)$. We assume $\dim P_\Lambda \mathcal{H} = n < N = \dim \mathcal{H}$.

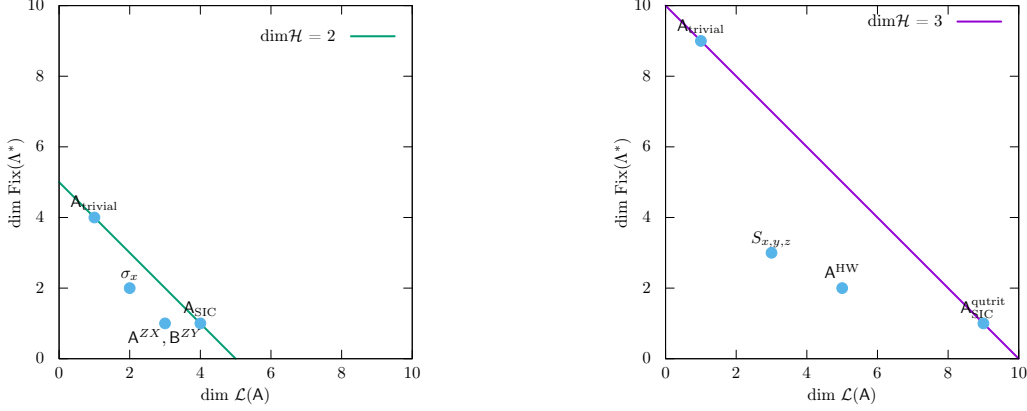


FIG. 3. Relation between $\dim \mathcal{L}(\mathbf{A})$ and $\dim \text{Fix}(\Lambda^*)$ in qubit and qutrit.

We can see that a map $\Lambda(P_\Lambda \cdot P_\Lambda)$ is well-defined as a channel on $\mathcal{S}(P_\Lambda \mathcal{H})$. In fact, as Lindblad has shown there exists an invariant state ρ_0 for Λ whose support projection coincides with $P_\Lambda \mathcal{H}$. By using this state ρ_0 , we observe that $\text{tr}[(\mathbb{1} - P_\Lambda)\Lambda(P_\Lambda \rho_0 P_\Lambda)] = \text{tr}[(\mathbb{1} - P_\Lambda)\Lambda(\rho_0)] = 0$. As any state σ on $P_\Lambda \mathcal{H}$ is dominated by ρ_0 , we conclude $\Lambda(P_\Lambda \sigma P_\Lambda) = P_\Lambda \Lambda(P_\Lambda \sigma P_\Lambda) P_\Lambda$ holds.

Let \mathbf{A} be an observable compatible with Λ . Then an observable $P_\Lambda \mathbf{A} P_\Lambda$ on a subspace $P_\Lambda \mathcal{H}$ is compatible with a channel $\Lambda(P_\Lambda \cdot P_\Lambda)$. We apply the first part of the present proof to $\Lambda(P_\Lambda \cdot P_\Lambda)$ to obtain

$$\dim P_\Lambda \mathcal{L}(\mathbf{A}) P_\Lambda + \dim \text{Fix}(\Lambda(P_\Lambda \cdot P_\Lambda)) = \dim P_\Lambda \mathcal{L}(\mathbf{A}) P_\Lambda + \dim \text{Fix}(\Lambda) \quad (8)$$

$$\leq n^2 + 1. \quad (9)$$

As $\mathcal{L}(\mathbf{A})$ is decomposed as $\mathcal{L}(\mathbf{A}) = P_\Lambda \mathcal{L}(\mathbf{A}) P_\Lambda + P_\Lambda \mathcal{L}(\mathbf{A})(\mathbb{1} - P_\Lambda) + (\mathbb{1} - P_\Lambda) \mathcal{L}(\mathbf{A}) P_\Lambda + (\mathbb{1} - P_\Lambda) \mathcal{L}(\mathbf{A})(\mathbb{1} - P_\Lambda)$, the dimension of $\mathcal{L}(\mathbf{A})$ is bounded by $\dim P_\Lambda \mathcal{L}(\mathbf{A}) P_\Lambda + (N^2 - n^2)$. Combining this bound with (9) ends the proof. \square

This theorem includes a well-known fact²² that an identity channel is compatible only with trivial observables. Interestingly, this theorem gives another proof of Theorem 1.

We plot $\dim \mathcal{L}(\mathbf{A})$ and $\dim \text{Fix}(\Lambda)$ for Lüders channels except for \mathbf{A}^{HW} in Figure 3. \mathbf{A}^{HW}

represents a channel defined by Kraus operators

$$K_1 = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad K_2 = \frac{1}{10} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sqrt{10} & 0 \\ 0 & 2\sqrt{10} & 0 \end{pmatrix}, \quad K_3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K_4 = \frac{1}{10} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\sqrt{10} & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}, \quad K_5 = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and an observable defined by $\mathbf{A}^{\text{HW}}(i) = K_i^* K_i (i = 1, \dots, 5)$. This channel does not satisfy property F¹⁹.

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